

# Mental Arithmetic

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# Table of Contents

## **Introduction**

Why mental arithmetic?  
Implementation of the mental arithmetic program

## **Mental Arithmetic**

Addition  
Subtraction  
Multiplication  
Division

## **Appendix I -- Using Manipulatives**

Chisenbop  
Proportional and non-proportional manipulatives  
Place value charts

## **Appendix II – Overheads, handouts**

Addition/Subtraction Chart  
Multiplication/Division Chart  
Place value charts  
Multiplication grids  
Chisenbop – Fingermath

## Introduction

### Why Mental Arithmetic?

At a time when emphasis in mathematics is conceptual learning and understanding mathematics, when we are moving towards increased problem solving and decreased drill and practice, and when students are allowed to use calculators on most mathematics tests starting in the middle grades, why spend time on mental arithmetic? There are many answers to this question, and as you will see when using this book, most of them are implied in the paragraph above.

#### Mental arithmetic emphasizes conceptual learning of mathematics

Mental arithmetic as taught here goes beyond drill and practice, and in some ways replaces drill and practice with exercises that are meaningful in a mathematical sense. All of the exercises presented in this book replace rote memorization with learning based on an understanding of the underlying logic of mathematics.

While learning addition, subtraction, multiplication and division facts, for instance, students learn about the properties of these operations to facilitate memorizing the tables. They apply the commutative property of addition and multiplication, for example, when they learn that  $3 + 7 = 7 + 3$  or that  $3 \times 7 = 7 \times 3$ . Knowing this greatly reduces the number of facts that need to be memorized. They use the distributive property when they learn that  $12 \times 7 = (10 + 2) \times 7 = 10 \times 7 + 2 \times 7 = 70 + 14 = 84$ .

All of the exercises place heavy emphasis on understanding the base ten system of numeration. At all levels, beginning with single digit addition, the special place of the number ten and its multiples is stressed. In addition, for instance, students are encouraged to add to ten first, and then add beyond the ten. Addition of ten and multiples of ten is emphasized, as well as multiplication by 10 and its multiples. Teachers are encouraged to teach these facts by using place value charts and manipulatives, again stressing a conceptual understanding of the process.

Connections between numbers are used to facilitate learning. The more connections that are established, and the greater the understanding, the easier it is to learn the many facts that need to be memorized in mathematics. It is the difference between memorizing nonsense syllables and learning a poem of the same length. In multiplication, for instance, students learn that they can get to  $6 \times 7$  if they know  $5 \times 7$ , because  $6 \times 7 = 5 \times 7 + 7$ . This kind of thinking provides students with confidence and a level of independence in mathematics. If they have forgotten a fact, they can reproduce it, often in several different ways.

In the approach to learning mathematics facts advocated here, time that is often devoted to rote learning of endless numbers of facts is instead used to help students understand the relationship between these facts. At the same time that the facts are memorized, students become “nimble” with numbers.

Teachers sometimes tell me that they could never do some of the computations in their heads that are included in this book, for instance the 2-digit multiplication. This does not mean that students won't be able to learn it, some quite easily. I therefore encourage teachers to teach even skills they think of as difficult; with some practice, students often become good at these skills. Students are not set in their ways as we are, and they should have an opportunity to learn even what we consider difficult. It also helps if we don't tell them that what they are learning is hard for us.

### Why mental arithmetic in an age of calculators?

In most classrooms I visit, mental arithmetic is not practiced. Students do all problems beyond simple single digit operations either by using the long written algorithm or by using a calculator. Why then is mental arithmetic useful? Again, there are many reasons. Let me list the ones that come to my mind, and there are probably others.

Mental arithmetic is faster than using a calculator or executing the written algorithm. The numbers dealt with in this book, on the whole, do not go beyond 2-digit length, and facility with computing these kinds of results mentally speeds up the problem solving process. It also enables students to focus on more difficult aspects of the problems that they need to solve. I have seen high-school students use a calculator to multiply by 10 or 100, and I frequently see elementary-school students solve the simplest problems by using the written algorithm of column addition or subtraction. Ease with the computation of relatively simple facts will give students an edge, even during tests where calculators are allowed.

But mental arithmetic is also a skill that is necessary in estimating answers, and the ability of estimating results is a prerequisite to using a calculator. Students need to have some ideal of the range of possible answers to avoid inadvertent miscalculations when using a calculator. While students should be encouraged to use a calculator and to focus their attention on learning problem solving skills, answers obtained through use of the calculator should always be checked mentally to verify that a mistake was not made in entering numbers and operations into the calculator.

### Mental arithmetic skills make math fun

Finally, mastery of mental arithmetic skills gives students a sense of accomplishment, a sense of ease with mathematics, which makes mathematics enjoyable. I have often seen students want to continue doing math problems after they learned a new skill that made a previously difficult task easy. Having watched students count up or down on their fingers or make chalk marks on the board to subtract 9, for instance, I showed them how to subtract 10, then add one. Many of them continued to ask for additional problems well into recess. Math learning can be fun when connections are made and when it leads to understanding.

### **Implementing the Mental Arithmetic Program**

The mental arithmetic program outlined here is easily integrated into the mathematics curriculum with any textbook series. It is an approach to learning number facts and computing numbers that supplements the regular curriculum. Teachers are encouraged

to integrate these units at the appropriate time when the skills are taught in the curriculum. Where students are still having difficulties with facts and concepts that were taught earlier, these skills should be re-taught using this approach.

Once a skill has been taught, it is important to keep practicing it. I recommend no more than five minutes of practice at a time, every day, choosing two or three of the skill boxes in this book each time. This insures that a variety of skills are practiced continuously. Once a skill is completely automatic, it can be dropped from the repertoire. It should be noted that all of these simple skills are practiced also in the context of other problems, such as word problems. Where necessary, the appropriate procedures and steps can be reviewed occasionally in this context as well.

In general, it is the frequency rather than the length of practice that fosters retention. Thus, daily brief practices of skills are most likely to lead to success. Similarly, students should be encouraged to take home the homework sheets provided for each of the lessons. These are grouped by approximate grade level, with each sheet including problem types of similar difficulty levels. Ask students to practice while watching TV, during commercial breaks – the mute button is perfect for this purpose. This provides students with the kind of short, frequent practice periods. It also takes no real time. Each practice sheet provides a brief review of several mental arithmetic skills, and a set of problems to practice.

### **School-Wide Adoption**

School-wide adoption of this program provides the greatest consistency for students. Students who are on level and who have used this approach from first grade should master all these skills by fifth grade. But in schools where students still experience difficulties with simple number facts and with understanding place value, this program can be used in the upper grades for these students.

Teachers need to use discretion in using this approach when students already have mastered some of these skills. This applies primarily to the simple addition, subtraction, multiplication and division facts with single-digit numbers. Once a student has memorized these facts, it does not make sense to teach the “tricks,” for instance using the knowledge of the fives table to multiply by 6. In other words, it is not necessary to re-teach a skill that has been learned in a different way by using this approach.

On the other hand, most students can benefit from the more difficult problems, those involving more than one digit. This is true even if they know the written algorithm and can solve these problems. The emphasis here is on mental arithmetic. It is also on understanding the place-value logic involved in the algorithms, as mental arithmetic replicates many aspects of the written algorithm. In other cases, as in multiplication by 5 (multiply by 10 and divide by 2), the skills involved are useful for numbers of all sizes.

A few teacher workshops will suffice in introducing these materials to teachers. The book is self-explanatory. Once teachers are introduced to the basic ideas of this approach, they will be able to pick the appropriate materials for their grade and the level

of their students. It is useful to have the math coordinator of the school or a teacher well versed in mathematics take the lead in helping teachers implement this approach.

Once teachers start using these materials with students, they will usually continue. Students enjoy these activities, and using the finger math or Chisenbop to have all students respond to problems encourages all students to participate and keeps them engaged. When used consistently and with understanding, both teachers and students experience success, and this tends to promote continuation of the program.

{A videotape that shows teachers explain and use these materials in the classroom is available to assist in the school-wide implementation of this program.

# Mental Arithmetic

## Addition

The following is a sequence of mental arithmetic activities related to addition. Teachers should select activities appropriate for the grade level; however, if students are not familiar with the simple procedures at the beginning, these should be taught, even at older grade levels. These procedures (including addition up to 10) are important for later ease in adding larger numbers.

In all of these activities, the emphasis is on making connections between diverse mathematics facts. This facilitates memorization, and makes retention of facts easier than rote memorization. In the end, this method reduces the infinite number of different math facts to a manageable group of related facts.

It is important that students engage in some form of mental arithmetic activity on a daily basis. This activity usually should not exceed five minutes, although teaching some of the initial concepts will take longer. Activities in the boxes are samples teachers can use for daily practice. Activities should be rotated, so that all skills are practiced regularly.

In any case, mental arithmetic exercises should form a separate part of the curriculum, and do not have to be tied in with current lessons. It is important that the activities are varied, and that concepts taught previously keep reappearing in the five minute daily mental arithmetic drills. In general, after a concept is first introduced, drills should focus heavily on the newly acquired skill. But soon, other previously learned skills should be included to embed them firmly in memory.

Although the teacher may have items written down for each five-minute session ahead of time, it is not necessary to duplicate these items. This is mental arithmetic, and students should put away all paper and pencils. Some information may be written on the board by the teacher to help students remember numbers in some problems; but students should not be allowed to do any writing during this period.

## **Checking student responses**

Mental arithmetic is most effective when all students participate simultaneously and the teacher has a way of checking all students' answers. Although there are different ways of doing this, for instance by having students write on small individual chalk boards, the best way is the use of Chisenbop. This is a method that allows students to represent numbers up to 99 on their two hands. There is more information on this method in the appendix. Even if Chisenbop is used for no other purpose than to give responses to mental arithmetic problems, the small investment in time to teach it is worth it.

Here are the basic principles of Chisenbop:

- The right hand represents ones, the left hand represents tens. To remember which hand is which, think about writing a number – the ones place would be on the right, the tens on the left.
- The right thumb counts as 5, the left thumb counts as fifty. Each other finger on the right hand counts as one, on the left hand as 10.
- To start counting, hold up the index finger of your right hand. This signifies 1; index and middle finger held up are 2; index middle and ring finger are 3; index, middle, ring and little finger held up count as 4. To indicate 5, the four fingers are folded back, and the thumb is held up. Six is thumb and index finger; 7 is thumb, index finger and middle finger; 8 thumb, index and middle finger. To show 9, all fingers of the right hand are held up.
- To indicate 10, the right hand is put down, and the index finger on the left hand is held up. For multiples of ten, copy the pattern of the right hand, but multiply by ten. So the thumb is 50, thumb and index finger 60, etc.

Since much of mental arithmetic involves numbers up to 99, this method works well to encourage full class participation in mental arithmetic. During the few minutes of each daily drill, all students will be engaged; the teacher will be able to assess the responses of all students; and students who need help can get immediate feedback. The appendix will show you ways to introduce Chisenbop to your class – it is really easy, and students enjoy using their hands to do math. Chisenbop has another advantage – it is tied to base ten and incorporates place value. Some types of mental arithmetic problems (e.g. adding tens) benefit from this fact.

We will now start with a progression of activities that will assist students in doing mental arithmetic problems in addition.

### 1. Numbers below 10

	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>
<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>
<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	
<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>		
<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>			
<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>				
<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>					
<b>7</b>	<b>8</b>	<b>9</b>						
<b>8</b>	<b>9</b>							

We will first look at the part of the addition grid that includes only numbers below 10. The body of that grid contains 36 different addition facts, from  $1+1=2$  to  $8+1=9$ . Eight of those involve the addition of 1 (top row, shaded) which leaves 28 facts. Fortunately, 16 of those facts are

duplicates of each other (shaded area), which leaves 12 other facts.

a. The Commutative Property of Addition

We owe this happy circumstance to the commutative property of addition. It is one of the important principles we will use in mental arithmetic. It simply means that  $3+5 = 5+3$ . Only young students will need manipulatives to see this, but here are some ways to show this concretely:

- Cuisenaire rods – a green and a yellow have the same length as a yellow and a green
- Stacking cubes – make two rods using different colors for the 3 and 5; hold them end to end with the 3 and 5 at opposite ends

We will use this principle of addition to help us with mental arithmetic (so go ahead, teach the big words to your class). It doesn't matter which number you start with, and it's easier to start with the larger number. So instead of adding  $3+5$ , have students add  $5+3$ . This is especially true of younger students who are still counting up to get the answer.

***Drill 1 – Start with the larger number, add the smaller one***

Using the commutative property of addition, have students turn problems around to put the larger number first. Some examples:

$2 + 6$ (compute $6+2$ in your head)	$2 + 7$
$3 + 5$ (compute $5+3$ )	$3 + 6$ etc.

b. Number Families

Even if your students are older, it will be worth a little time to introduce them to the number families up to ten. A number family is made up of all the pairs of numbers that add up to the same sum. Here are some number families:

$$4 = 0+4, 1+3, 2+2, 3+1$$
$$5 = 0+5, 1+4, 2+3, 3+2, 4+1$$
$$6 = 0+6, 1+5, 2+4, 3+3, 4+2, 5+1 \text{ etc.}$$

In the younger grades, manipulatives are useful to reinforce this concept. Use Cuisenaire rods, and have students find all pairs of rods that can be combined to make 8 etc. Stacking cubes can be used the same way, first making a train of length 8, then using two colors and making trains that show  $1+7$ ,  $2+6$  etc.

***Drill 2 – Number Families***

Write a number on the board to show a number family. Then say a number, and have students respond with the appropriate number that completes the sum.

Example: Write 8    Say 2; students say 6

5	3
7	1

Keep going for a while, then write a new number on the board.

Addition families are important for two reasons: They tie addition to subtraction, and they help in the next activity, in which students add beyond 10. Before we go to that activity, however, the number 10 needs to get its special place in the number family album.

***Drill 3 – Number Families -- 10***

Students use Cuisenaire rods or stacking cubes to represent the number family 10. Then use the same drill as in #2, writing 10 on the board. Adding up to ten must become automatic, so this drill should be repeated often until all of these facts are learned.

Write 10 on the board; you say a number, students say the number that adds up to ten:

Teacher:	2	Students	8
Teacher:	7	Students:	3 etc.

**3. Numbers Above 10**

We now turn to the addition table on the next page, which includes 100 addition facts. To memorize this table as a set of unrelated facts is overwhelming, so we will reduce this task by applying a few principles.

- a. Applying the commutative property, we have reduced this chart to 55 numbers – the shaded area in the table indicates addition facts that are repeated like  $6+7$  and  $7+6$ . Remind students to start with the larger number, and to compute  $7+6$  rather than  $6+7$ .
- b. We can cross off column 1, as adding by one is simply counting up by one, and surely all students already know how to do this. So now we are down to 45 numbers to learn.
- c. We might consider crossing out the last row, as adding 10 to each number would appear easy. However, students do not automatically know that 10 is a special number in our system of numeration, and so we need to make a big deal of this. I often see students count up on their fingers to add 10, thus adding ten ones. It is much easier to add one 10. It also helps students get a better understanding of our place value system, which is based on 10.

Imagine students thinking of each number as completely distinct, like a whole lot of pictographs that each stand for a specific number. In this way of thinking, there are no patterns. Each set of facts needs to be memorized separately. Just as we have to memorize  $7+8$ , we need to memorize  $10+8$ . I have met many students whose view of mathematics is just exactly like that. No wonder they have a hard time memorizing their facts. There are so many of them!!

Learning the patterns that reduce mathematics to a more manageable system begins with emphasizing the importance of 10, which is shown in the last row of the addition table. Indeed, once we understand our place value system, we do not have to memorize that row.  $10+1=11$  (which is simply another word for ten-one); twelve is ten-two; thirteen is ten-three; this becomes more obvious with fourteen. In first grade, teachers should use the words “ten-one,” “ten-two,” and “ten-three” interchangeably with 11, 12 and 13 to make this point.

***Drill 4 – Adding 10***

This drill is important mainly in first grade – it will be repeated later for larger numbers in older grades. It is necessary for young students to understand that 11, 12, 13 etc. Are  $10+1$ ,  $10+2$ ,  $10+3$ . In the drill we reverse the numbers, and have students solve  $3+10$ ;  $5+10$ ,  $1+10$  etc.

Once students understand the special significance of the number 10, and are no longer tempted to add ten ones rather than one ten, we can eliminate the last row of the addition table. It is easy to remember.

We are down to 35 numbers that need to be remembered. We have already learned the eleven numbers that add to less than 10, and so we are left with memorizing 24 addition facts now, all of them resulting in a 2-digit number up to 19 (we’ll deal with  $10+10$  as a special fact).

***Simplifying addition by adding to 10, then adding on***

This is a procedure for adding single digit numbers that has several advantages, although at first it may seem cumbersome. It reinforces the special status of the number 10; and it will allow students to transfer the addition skill from single digit numbers to multiple digit numbers. Thus, a pattern learned now will enable students to solve more difficult problems later. Note that this process requires familiarity with number families that has been acquired earlier.

Example:  $7+5$ :

It takes 3 to get from 7 to 10; we have added 3, so there is 2 left to add ( $5-3=2$ ), and solve the problem in two steps:  $7+3 = 10$ ;  $10+2 = 12$ .

Example:  $3+8$ :

Remember to use the commutative property of addition first to put the larger number first. So this becomes  $8+3$ .  $8+2=10$ ;  $3-2=1$ ;  $10+1=11$

***Drill 5 – Single digits beyond 10 – two step process***

$8+5$	$6+5$	$7+7$	$9+4$	Put small number first, reverse:
$7+6$	$8+7$ etc.			$3+9$ $5+8$ $4+7$ $2+9$ etc.

#### 4. Doubles

It is worth having students memorize the “doubles.” This is easy to do – there are only 10, two of them already shaded gray in the chart (1+1 and 20+20), showing that they have already been learned. So there are only eight doubles to remember, and most children will already know those. You may want to point out that all doubles are even numbers. Even numbers are numbers that are multiples of 2, are divisible by 2. If you group an even number of objects into pairs (twos), there will not be one left over. You can show this even to first grade students using any kind of counters. If you divide 5 chips into pairs, you have one “odd” one left over. This does not happen for 4 or 6.

##### ***Drill 6 -- Doubles***

1+1, 2+2, 3+3, 4+4, 5+5, 6+6, 7+7. 8+8, 9+9, 10+10 – mix them up...

#### 5. Adding 10 and multiples of 10

For students who can not do the mental arithmetic involved in adding tens and multiples of tens to two-digit numbers, it is necessary first to start with concrete manipulatives. So this is a brief introduction to just a few of the materials that can be used prior to introducing mental arithmetic problems. Additional information on using manipulatives for teaching place value is in Appendix II.

##### ***Drill 7 – Addition of 10 and multiples of 10 Numbers up to 100***

25 + 10	88 + 10	12 + 20	15 + 80
79 + 10	etc.	54 + 40	6 + 90
5 + 10		23 + 30	etc.
27 + 10		27 + 70	

You may want to go the next step – adding multiples of ten large enough to go beyond 100 with numbers such as 60+50. Again, lay the groundwork with the place value chart, and show that 6 tens + 5 tens makes 11 tens, or 110. Have students practice with the unit digit being zero at first, then go to problems like 69+50 = 119.

##### ***Drill 8 – Addition of 10 and multiples of 10 Numbers over 100***

30 + 80	33 + 80	80 + 70
70 + 50	76 + 50	89 + 70 etc.

Use the same preparation as above, including the place value chart. When dealing with numbers larger than 100, it will be necessary to switch to non-proportional manipulatives, where a chip or cube in the 100's column represents 100. Go to mental arithmetic once the foundation is laid in this way. You may want to write the first number on the board.

**Drill 9 – Addition of multiples of 10 to larger numbers  
Addition of 100 and multiples of 100**

523 + 60	394 + 100
1,785 + 10	529 + 400
6,869 + 30	3,651 + 300
285 + 40	5,873 + 600 etc.

**6. Adding 9, 19, 29, 39 etc.**

The number nine has a special place in the decimal system, because it is equal to 10-1. (A separate section deals with the “magic of nine.”). Thus, we can apply what we have learned about adding tens to adding numbers that end in nine. For instance,  $25 + 9 = 25 + 10 - 1 = 35 - 1$ . The same principle applies to other numbers ending in nine. We will practice only numbers that add to 100 or less here.

To prepare students for doing these problems as a mental arithmetic exercise, it is useful to first use the manipulatives to be sure they fully understand the principle involved. Of course some students will not need this extra step. But for those who do it is important. It is the difference between rote memorization and remembering from principle, which is much easier to do.

Two types of manipulatives are recommended: Place value mats (with proportional or non-proportional manipulatives -- bundles or chips); and Chisenbop or fingermath. The latter enables children who need it to always access this “crutch.”

For students who have thought of nine as a difficult number, counting up on their fingers or making marks to count by to nine, this is a liberating experience.

**Drill 10 – Adding 9, 19, 29 etc.**  
(Note: add 10-1, 20-1, 30-1 etc.)

27 + 9	23 + 9	15 + 9
27 + 19	33 + 29	69 + 19
27 + 39	53 + 39	23 + 59
27 + 69	73 + 19*	45 + 29 etc.

\*start with this pattern first. then continue with mixed numbers

## 7. Adding any two digit number

Once you have come this far, students will be able to apply what they have learned to do addition of two-digit numbers by mental arithmetic. They can do column addition using paper and pencil for any numbers larger than this. You may want to write the first number on the board, but give the second number orally. In most situations where they have to do arithmetic they will usually have both numbers in writing (as on standardized tests). The ability to solve the problems mentally merely gives them an edge of added security and speed.

As you do these problems, remind students of some of the “tricks” they have learned previously.

### ***Drill 11***

#### ***Adding 2-digit numbers***

$45 + 23$

$76 + 13$

$28 + 61$

$56 + 37$

$28 + 49$

$24 + 67$

## Subtraction

	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	12
3	4	5	6	7	8	9	10	11	12	13
4	5	6	7	8	9	10	11	12	13	14
5	6	7	8	9	10	11	12	13	14	15
6	7	8	9	10	11	12	13	14	15	16
7	8	9	10	11	12	13	14	15	16	17
8	9	10	11	12	13	14	15	16	17	18
9	10	11	12	13	14	15	16	17	18	19
10	11	12	13	14	15	16	17	18	19	20

Subtraction and addition are closely related, and many of the exercises used to practice addition will also be useful in subtraction. Again, first we look at numbers up to 10, and at the number families practiced in addition. As in addition, there will be extra emphasis on the number 10.

The table above is, of course, the addition table. It can also be used as a subtraction table. The process is illustrated using the number sentences  $10-3=7$  and  $10-7=3$ . Going along the diagonal (highlighted) is the number family for 10:  $10-1=9$ ;  $10-2=8$ ;  $10-3=7$  etc. Students can use the “pointer” in the Appendix to help with the subtraction facts in the table. Note that the sums for all number families are on a diagonal (the sevens are also highlighted). This is another way to look at number families.

To highlight the connection between addition and subtraction, students should learn the “missing addend” function of subtraction at this point. In the example above, we could rewrite the equation  $10-7=3$  as  $7 + a = 10$  (seven plus how many is ten). In both cases the number to solve for, the “unknown” is 3. (Note that there are many subtraction word problems that require knowledge of the “missing addend” use of subtraction. Examples: there are 10 children, 7 are girls. How many are boys? Bill wants to buy a book that costs \$10; he has \$7. How much money does he need before he can buy the book? Appendix)

#### **Drill 12 -- Number Families – Subtraction (Sums up to 9)**

To drill a number family, write the sum on the board. Then call a number, and let the students find the missing addend. Examples:

Write **9** on the board; then call numbers to 8; students respond with missing addend. Write **6** on the board, call numbers to 5 etc.

**Be sure to show students that this is equivalent to subtraction: in the first example,  $9 - 1 = 8$ ; in the second,  $5 - 1 = 4$  etc. Practice problems in this way, as well.**

#### **Drill 13 -- Number Families – 10**

Write 10 on the board. Then call a number up to 9; students respond by giving the number that adds to 10.

#### **Drill 14 – Subtracting the ones from 2-digit numbers**

In this exercise, all ones are subtracted from 2-digit numbers. This is easy, but needs to be mentioned specifically.

$13 - 3$        $27 - 7$        $18 - 8$        $95 - 5$        $56 - 6$        $22 - 2$       etc.

These drills set the stage for subtraction with regrouping. When doing mental arithmetic, this will be taught as a two-stage process, for example:  $15 - 7$ ;  $15 - 5 = 10$ ;  $10 - 2 = 8$ . First the ones are subtracted, so we get to 10 (or another multiple of 10). It is now important for students to see immediately that  $7 = 5 + 2$  (number families drill); after the five ones have been subtracted, two more must be subtracted from 10. The tens-family facts are also important at this point.

Practicing this two-step process is important for several reasons: It reduces the number of facts students need to memorize, as knowledge of the number families is all that is required. Also, the two-step process works equally for multiples of 10, not just for numbers in the teens.

**Drill 15****Subtraction from “teens” to below ten**

$13 - 8 (13 - 3 - 5)$

$18 - 9 (18 - 8 - 1)$

$12 - 6 (12 - 2 - 4)$

$15 - 7 (15 - 5 - 2)$

$16 - 8 (16 - 6 - 2)$

$14 - 9 (14 - 4 - 5)$

etc.

From this drill, we move to larger 2-digit numbers. It is useful to show students how this process corresponds to the column-subtraction algorithm they have learned. In fact, they are subtracting with trading or borrowing:

$$\begin{array}{r} 5 \\ 6 \overset{1}{5} \\ - 7 \\ \hline 5 \ 8 \end{array} \longrightarrow 65 - 5 = 60; 60 - 2 = 58$$

**Drill 16****Subtraction with trading – 1-digit numbers from 2-digit numbers****Use 2-step process**

$73 - 5$

$25 - 6$

$44 - 8$

$98 - 9$

$24 - 7$

$41 - 3$

etc.

The next step in practicing subtraction as mental arithmetic is the subtraction of tens. Again, this needs to be taught explicitly. At the beginning various manipulatives reinforcing place value should be used. These are described in the appendix. One of these “manipulatives” is Chisenbop, or finger math (see appendix). In this system, the fingers of the left hand constitute tens (with the thumb representing 50). The value of using manipulatives first cannot be overemphasized. Even for older students who have not benefited from using hands-on materials, it will be useful to use bundles, base-ten blocks etc. Prior to doing the mental arithmetic exercises.

**Drill 17****Subtracting 10 from 2-digit numbers**

$54 - 10$

$75 - 10$

$33 - 10$

$15 - 10$

etc.

**Drill 18****Subtracting Multiples of 10**

$54 - 40$

$75 - 50$

$33 - 20$

$68 - 30$

etc.

Now we are ready to tackle subtraction of two-digit numbers from two digit numbers. The first exercise will involve numbers that do not require regrouping.

**Drill 19**

**Subtraction of 2-digit numbers without regrouping**

54 – 43      75 – 52      37 – 25      68 – 34      49 – 56      82 – 21      99-38  
etc.

Next we will do subtraction with regrouping. It is important to remind students of all that has been learned. Encourage students to first subtract the ones, using the tw-step process, and then to subtract the multiples of ten. For example,  $53 - 27$  is solved mentally as follows:  $53 - 3 = 50$ ;  $50 - 4 = 46$  (two-step process);  $46 - 20 = 26$ . Model these steps before starting the drill.

**Drill 20**

**Subtraction of 2-digit numbers with regrouping**

54 – 46      75 – 58      37 – 29      62 – 34      43 – 57      82 – 29      91-33  
etc.

Next we will duplicate the exercises on adding nines by subtracting nines and numbers ending in nines. We show students that subtracting 9 is equivalent to subtracting 10, then adding back the additional one that was subtracted. For example,  $17 - 9 = 17 - 10 + 1$ . We do this because tens are easier to handle in our base-ten place-value system.

**Drill 21**

**Subtraction of nines (subtract 10 and add 1)**

54 – 9      75 – 9      37 – 9      62 – 9      43 – 9      82 – 9      91-9  
etc.

Now we are ready to subtract any 2-digit number that ends in nine. We first round the number to the next multiple of 10, subtract that multiple of 10, then add one. For example  $63 - 29 = 63 - 30 + 1$ .

**Drill 22**

**Subtraction of numbers ending in 9**

74 – 49      75 – 59      37 – 29      62 – 39      43 – 19      82 – 59      91 - 69  
etc.

## Subtraction by Adding

One use of subtraction is in finding the missing addend, as in the number sentence  $5 + \quad = 8$ . To solve this problem, we usually convert the number sentence (equation) to  $8 - 5 = 3$ . But we can also think of it as finding out how much to add to 3 in order to get 8.

It is usually easier to subtract by adding from the smaller number (the subtrahend) up to the larger number the minuend). In doing so, we have the additional advantage of using the number facts learned previously in addition when working with number families. All families will be important, but, again, the family of tens is the most important. Doing subtraction by finding the missing addend has the additional advantage of avoiding the issue of trading or borrowing.

Let us solve  $15 - 7$ , for example. We convert this problem to  $7 + \quad = 15$ . We then divide the problem into two steps: It takes 3 to add up to ten; and it takes five more to add up to 15. The answer is the sum of the two numbers,  $3 + 5$ . The following drill has items in increasing levels of difficulty. The first column results in one-digit answers; the second has two-digit answers. The third column is identical in concept to the first two, with larger numbers, but differences that remain in the teens (the sum of two one-digit numbers).

### **Drill 23**

#### **Subtracting by adding**

$6 + \quad = 13$	$3 + \quad = 17$	$25 + \quad = 34$
$8 + \quad = 17$	$5 + \quad = 19$	$52 + \quad = 65$
$5 + \quad = 11$	$7 + \quad = 18$	$73 + \quad = 88$
$3 + \quad = 12$	$4 + \quad = 16$	$37 + \quad = 49$

(Also drill as  $13 - 6$  etc.)

The drill above is useful for students who are having difficulty with the subtraction part of long division problems, and may be introduced at that time.

In the next drill, we carry this process one step further. We will stay with two-digit numbers, but the numbers will not be adjacent tens, as in the previous drill. This will add an additional step to the addition problem, as illustrated below:

$$26 + \quad = 53; 26 + \mathbf{4} = 30; 30 + \mathbf{20} = 50; 50 + \mathbf{3} = 53$$

Add  $\mathbf{4} + \mathbf{20} + \mathbf{3} = \mathbf{27} \rightarrow$  the answer

**Drill 24**  
**Subtracting by adding**

$16 + = 53$	$23 + = 57$	$25 + = 84$
$58 + = 87$	$45 + = 79$	$52 + = 75$
$25 + = 41$	$67 + = 88$	$73 + = 98$
$33 + = 72$	$74 + = 96$	$37 + = 69$

(Also drill as  $53 - 16$  etc.)

Subtracting by adding also facilitates subtraction of larger numbers. As an example, we will subtract 2-digit numbers from numbers in the hundreds. In this case, students first complete (add to) 100. They may do this in two steps, but will learn quickly to do it in just one. In the example below, students may add from 85 to 90 first (5), then add to 100 (10), then add to 123. In fact, they are doing the “100” family as the previously did the families, and will learn that  $10 + 90 = 100$ ,  $20 + 80 = 100$ ,  $30 + 70 = 100$  etc.

$$123 - 85 \longrightarrow 85 + 15 = 100; 100 + 23 = 123; 15 + 23 = 38, \text{ the answer.}$$

**Drill 25**  
**Subtracting by adding: Numbers over 100**

$112 + = 69$	$112 - 69 =$
$145 + = 67$	$145 - 67 =$
$127 + = 58$	$127 - 58 =$
$152 + = 89$	$152 - 89 =$

Daily Instruction

When you have gotten this far, be sure to repeat the drills regularly. Choose one or two drills daily, for no more than five minutes each time. Be sure to repeat the principles involved if necessary. Once students are doing the mental arithmetic using the principles described here, it is possible to include more different types of drill in each session. The more frequently each type of problem is drilled, the more likely students are to remember that kind of problem.

When drilling, it is important, however, that students recognize the problem type, so that several of the same type of problem should be given. The procedures outlined here work because they teach addition and subtraction facts in terms of a few principles; students learn to do arithmetic by remembering these principles, rather than remembering individually the hundreds of facts that are involved. As these drills are repeated, students will simply remember some of the number facts without using the

steps described here, and that is OK. However, emphasis in these drills should always be on learning the principles involved, not on memorizing individual number facts, particularly when 2-digit numbers are involved.

This is not to say that number facts are unimportant. If there is regular (i.e. brief daily) practice of mental arithmetic, the necessary number facts will be easily recalled by all students. Procedures that are first drilled as multi-step problems will become single step problems in students' minds. However, even if this is not true, and if students only remember the minimum number of facts instantly (the facts associated with the number families up to 10), the steps are not too cumbersome. If the process makes sense to students, it will also be easily remembered. This is why it is important to explain every step, if necessary through the use of manipulatives.

### Alternate Algorithm for Column Subtraction

*The following algorithm for column subtraction is taught in some countries. It is in some ways simpler than the algorithm commonly taught in American schools, particularly when there are zeroes in the minuend, the number on top. The algorithm relies on the principle that the difference between two numbers remains the same even when the same constant is added to each number. Thus,  $10 - 2 = 8$ ;  $11 - 3 = 8$ ;  $12 - 4 = 8$ , etc. This is not such a difficult principle to grasp, and most students will easily understand it. If they have siblings, they will understand that the differences in their ages remains constant, even though they are all getting older.*

Here then is an example of this algorithm, in which tens, hundreds etc. are added to each column, rather than borrowed. The first example does not include any zeroes in the minuend. This does not show the advantages of this algorithm, but it will be easier to understand the principle.

$$\begin{array}{r}
 263 \\
 - 28 \\
 \hline
 \end{array}
 \qquad
 \begin{array}{r}
 26^13 \\
 - 3\ 8 \\
 \hline
 23\ 5
 \end{array}
 \qquad
 \begin{array}{r}
 26\ 3 \\
 - 2^18 \\
 \hline
 23\ 5
 \end{array}$$

Again, we have added one ten to the minuend, the number at the top, then borrowed that ten to make the 13. Therefore, we also have to add ten to the bottom. In fact we are subtracting 38 from 273, which has the same result as subtracting 28 from 263.

Because we borrowed the 10, we don't have to actually add it to the 6 tens. However, we do have to add it to the two tens of the subtrahend, thus having to subtract 3 (tens) from 6 (tens). In the American algorithm, we would subtract 2 (tens) from 5 (tens), having borrowed one ten from the original 6 tens. The third column shows a way to represent this algorithm, with the one (ten) written to remind students to add it to the 2 (tens), as well as to show that a ten has been added to the 3 ones, which become 13.

The next example shows the advantages of this algorithm when the minuend has zeroes, a situation that is difficult for some students using the traditional algorithm.

$$\begin{array}{r}
 3,001 \\
 \underline{-1,237} \\
 \hline
 \end{array}
 \qquad
 \begin{array}{r}
 3,^{1}0^{1}0^{1}1 \\
 \underline{-2,347} \\
 1,764
 \end{array}
 \qquad
 \begin{array}{r}
 3,001 \\
 \underline{-1,237} \\
 1,764
 \end{array}$$

Clearly, this algorithm should not be introduced to students who have no problems with traditional subtraction. However, it might be of interest to gifted students as an alternative way of doing things. It is a common way of teaching subtraction in parts of Europe. It also may be useful to show this algorithm to students who have a hard time executing the traditional American subtraction algorithm.

## Multiplication

### The Multiplication Table

The table below is a multiplication table in which really easy numbers and duplicate numbers are shaded. Only the numbers in the white squares need to be memorized.

Although multiplication by 10 is easy, this must be practiced with students, preferably (in the early grades) by using a place value mat. Multiplication by 1 (the “identity property” of multiplication) also needs to be learned. Not included in the table but important to teach is multiplication by zero, which always results in zero.

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Note that the 100-cell matrix now has been reduced to 37 cells. This is because multiplication, like addition, is commutative:  $3 \times 7 = 7 \times 3$ . Students can choose which number is easier for them to remember. Since multiplication by 2 and 3 are usually learned first, students remember these facts first. Thus,  $7 \times 3$  may be easier than  $3 \times 7$ . On the other hand, if students still multiply by repeated addition (I have observed students doing this when solving long division problems, for instance), then  $3 \times 7$  is easier to solve.

## Multiplication by two

### **Drill 26**

#### **Multiplication by 2**

2x1 2x2 2x3 2x4 2x5  
2x6 2x7 2x8 2x9 2x10

(You will mix up the numbers).

Students have been counting by twos from first grade on, and multiplication by two should not be a problem. If they are having difficulties, show them how to solve the problem by addition (see chapter one, doubling numbers). Thus,  $2 \times 8 = 8 + 8 = 16$  (see the section on addition for more detail). Note that there are only 8 “hard” numbers to remember, as 3 of them have products under 10.

Multiplication by 3 requires learning 7 numbers, one less than the twos because  $2 \times 3$  is already known as  $3 \times 2$ . If students have trouble with multiplying by 3, they can refer back to the twos:  $7 \times 3 = 2 \times 7 + 7$  ( $7 \times 3 = 7 + 7 + 7 = 2 \times 7 + 7$ ). Only students who have not learned their threes table should use this crutch. However, it is better to teach a crutch to help students reproduce an answer than to have them look up the answer if they have not memorized it yet. If you practice a few minutes each day (and don't forget to include some addition and subtraction problems as well), they will eventually remember the answer without resorting to the crutch. Always remind students of the commutative property of multiplication when practicing the multiplication table: If they do not know  $2 \times 3$ , they can substitute  $3 \times 2$ , for instance.

### **Drill 27**

#### **Multiplication by 3**

1x3 2x3 3x3 4x3 5x3  
6x3 7x3 8x3 9x3 10x3

(You will mix up the numbers).

There are six new numbers in the fours table that students have to learn. If they have

### **Drill 28**

#### **Multiplication by 4**

1x4 2x4 3x4 4x4 5x4  
6x4 7x4 8x4 9x4 10x4

( $3 \times 4 = 4 \times 3$ )

not memorized these facts, teach them an easy trick for multiplying by four (which also helps with larger numbers later on):  $4 = 2 \times 2$ . So instead of multiplying by 4, students can multiply twice by 2. For example:  $4 \times 6 = 2 \times (2 \times 6) = 2 \times 12 = 24$ . This is due to the associative property of multiplication, where  $(2 \times 2) \times 6 = 2 \times (2 \times 6)$ . Another way of

saying this is that multiplying a number by four is the same as doubling the number twice: first double the 6, then double the 12.

At this point you may want to point out some of the patterns in multiplication that help students check their answers quickly for obvious mistakes. The product of any number multiplied by an even number (a multiple of 2) will be an even number. An odd number multiplied by an odd number will always result in an odd product.

There are five new number facts in the fives table (students already know the facts through  $4 \times 5$  if they know the twos, threes and fours table – remember the commutative property of multiplication where  $3 \times 5 = 5 \times 3$ ).

There is an easy trick to help in multiplying by five, which comes from the fact that  $5 = 10/2$ , that there are two fives in one ten. This trick will be helpful later in multiplying larger numbers by 5. Instead of multiplying a number by 5, we first divide it by 2 and then multiply by 10. Of course this is easiest with even numbers:  $6 \times 5 = 6/2 \times 10 = 3 \times 10 = 30$ . For odd numbers, multiply by the next lower even number and then add 5.  $7 \times 5 = 6/2 \times 10 + 5 = 3 \times 10 + 5$ . It is also possible to multiply by 10 first, then divide by 2. Is this “trick” easier than simply memorizing the fives table, which isn’t that difficult to remember? Perhaps not. But as in all of these shortcuts, connections are made between different mathematics facts, and these connections in themselves are useful for students.

**Drill 29**

**Multiplication by 5**

1x5   2x5   3x5   4x5   5x5  
6x5   7x5   8x5   9x5   10x5

(Note: for  $2 \times 5$ , think  $1 \times 10$ ;  $4 \times 5 = 2 \times 10$ ,  $6 \times 5 = 3 \times 10$ ,  $8 \times 5 = 4 \times 10$ ;  $4 \times 5 = 5 \times 4$ )

There are only four new numbers in the table of sixes, from  $6 \times 6$  to  $9 \times 6$ . An easy way to remember  $6 \times 6$  is as  $5 \times 6 + 6$  (five sixes plus one six); similarly,  $7 \times 6$  can be remembered as  $5 \times 6$  plus two sixes, or  $5 \times 6 + 2 \times 6 = 30 + 12$ . This leaves only  $8 \times 6 = 48$  and  $9 \times 6 = 54$  as numbers that have to be memorized. Some of these numbers will be revisited when we learn the eighths and nines tables. It may also help some children to know that  $6 = 2 \times 3$ ; so to multiply by 6 they can first multiply by 3, then multiply the result by 2. For example,  $7 \times 6 = 7 \times 3 \times 2 = 21 \times 2 = 42$ .

**Drill 30**

**Multiplication by 6**

1x6   2x6   3x6   4x6   5x6  
6x6   7x6   8x6   9x6   10x6

(Note:  $7 \times 6 = 7 \times 3 \times 2 = 21 \times 2 = 42$ ;  $4 \times 6 = 6 \times 4$ )

**Drill 31**

**Multiplication by 7**

1x7   2x7   3x7   4x7   5x7  
6x7   7x7   8x7   9x7   10x7

(Note:  $6 \times 7 = 7 \times 6$ )

The sevens and eights tables tend to be the hardest to remember for students. However, remember that there are few new number facts to remember, only three new facts for the sevens ( $7 \times 7$ ,  $8 \times 7$  and  $9 \times 7$ ), and only two for the eights ( $8 \times 8$  and  $9 \times 8$ ). The sevens numbers should be memorized, though the nines will receive special treatment later.

A special trick applies to the eights, since  $8 = 2 \times 2 \times 2$ , which is also  $4 \times 2$ . To multiply by 8, students can multiply by four and then double the number; or double the number twice. For  $8 \times 8$ , we get  $4 \times 8 \times 2 = 32 \times 2 = 64$ ; or  $2 \times 8 \times 2 \times 2 = 16 \times 2 \times 2 = 32 \times 2 = 64$ . Again, while students should memorize the table, this crutch can be helpful in allowing them to find an answer with relative ease if they don’t have instant recall. This will assist in instant recall in the long run.

**Drill 32**

**Multiplication by 8**

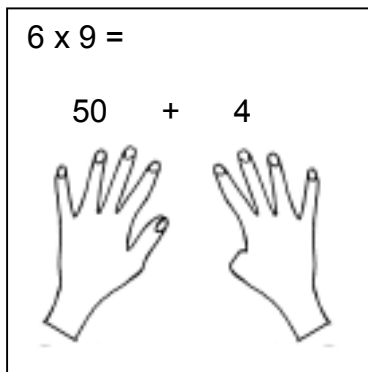
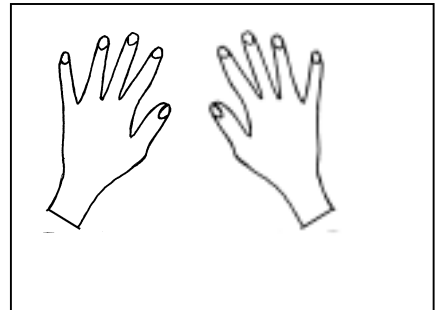
1x8   2x8   3x8   4x8   5x8  
6x8   7x8   8x8   9x8   10x8

(Note:  $3 \times 8 = 8 \times 3$ ;  
 $7 \times 8 = 7 \times 2 \times 2 = 14 \times 2 \times 2 = 28 \times 2 = 56$ )

This trick also helps later in multiplying larger numbers by eight mentally. Also, students have already memorized most of the facts in the eight's tables if they know the tables up to eight. This is due to the commutative property of multiplication.

The nines have a special place in our numbering system because of their proximity to 10. All tricks relating to multiplying by 9 come from the fact that  $9 = 10 - 1$ . We use the fact that it is easy to multiply by 10 to facilitate multiplication by 9. There are many ways this can be taught. One way is to first multiply by 10, then subtract the number that was multiplied. For example,  $6 \times 9 = 6 \times (10 - 1) = 6 \times 10 - 6$ . (Note that this illustrates the distributive property of multiplication and addition/subtraction).

There is a simple pattern that results, and this pattern can be taught. The tens digit is one less than the number multiplied by 9, and the ones digit is the difference between the number and 10. So in  $6 \times 9$ , the tens digit is  $6 - 1 = 5$ , and the ones digit is  $10 - 6 = 4$ , for a result of 54. If you add the ones and the tens digit, the result will always be 9. So you can use this fact to arrive at the ones digit: it is 9 minus the tens digit. In  $6 \times 9$ , the tens digit is 5;  $5 + 4 = 9$ , so the answer is 54.



Using your fingers to multiply by nine replicates this pattern. Place your hands in front of you as illustrated here. Now bend down the little finger of your left hand. This represents  $1 \times 9$ : The first finger is bent (representing the one), and there are nine fingers to the right of the bent finger, representing nine ones. There are no tens, as there are no fingers to the left of the bent finger. Now bend down the second finger of your left hand, keeping all other fingers straight. The second finger represents multiplication by 2; the fingers to the left of the bent finger are the tens, the fingers to the right are the ones. There is one ten (the little finger on the left hand), and there are 8 tens (the 8 fingers to the right of the left ring finger, which is bent under).

The diagram to the right represents  $9 \times 6$ : The sixth finger from the left is bent under (the thumb of the right hand). There are five fingers to the left representing tens, and four fingers to the right representing ones. The answer is 54.

<b>Drill 33</b>				
<b>Multiplication by 9</b>				
1x9	2x9	3x9	4x9	5x9
6x9	7x9	8x9	9x9	10x9

Also have students write down the nines facts in a column. They will see that the number in the tens column increases by one, from zero to 9; the number in the ones column decreases by one, from 9 to zero. This is the pattern replicated in using the fingers to multiply by nine.

## Multiplication by 10 and Multiples of 10

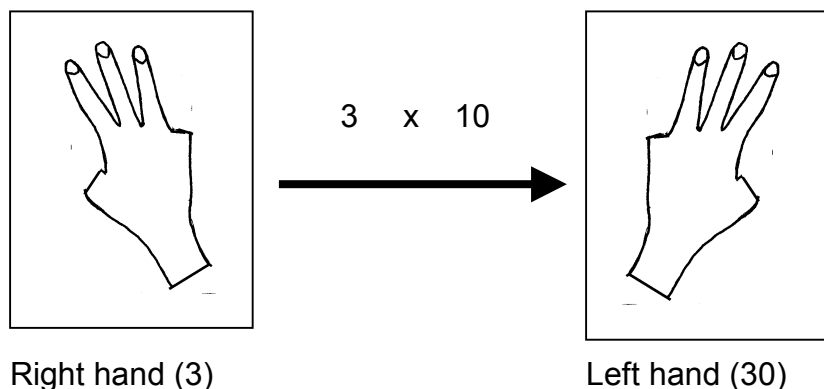
The importance of the number 10 in our base ten numbering system has to be continuously emphasized to students. Without this emphasis, multiplication by 10 (or multiples of 10) is just another group of unrelated math facts to memorize. With an understanding of 10 and its special role in our numbering system, multiplication by 10 or its multiples is easy. While this is an obvious observation to most of us, it is not obvious to children. I have seen high-school students use a calculator to multiply by 10.

For younger children, it is important to start with proportional manipulatives, objects that represent the relative size of ones, tens, hundreds and thousands. These manipulatives are combined with place value charts. Place value charts should be modified to reflect the size of numbers of each grade level. In first and second grade they should include ones, tens, and hundreds, then they can be expanded to include thousands. When students learn to represent larger numbers, place value charts representing numbers over one million are appropriate. Place value charts representing decimal values are also appropriate.

There are two major sets of proportional manipulatives: Base-ten blocks and materials that can be bundled (popsicle sticks, coffee stirrers, straws). It is good to work with more than one type of manipulative. Bundles have two advantages: materials are cheap and easy to obtain (bundles are made with rubber bands), and the process of “bundling” and “unbundling” is equivalent to the process of trading. As the process of making bundles of 10 and 100 can be cumbersome, children sometimes suggest skipping the process altogether, and letting one popsicle in the tens column represent a bundle of tens. This provides a transition to nonproportional manipulatives, where one object (any counting object will do: chips, beans, cubes) represents the value of the column it is placed in. A chip in the hundreds column represents 100, in the tens column it represents 10 etc. Nonproportional representation becomes necessary when dealing with large numbers or with decimals on the place value chart.

The first activity designed to help children understand multiplication by ten should involve the bundling of sticks and a place value chart with ones, tens and hundreds. Have children place one popsicle stick in the ones column. Explain that each one turns into one ten – have them make a bundle of ten, then show that multiplication by ten means replacing the one with a bundle of ten in the tens column. Repeat this activity with numbers greater than 1, such as 2, 3, etc. Then show how this translates into the written algorithm:  $1 \times 10 = 10$ : the blank ones column is represented by the zero, and the bundle in the tens column is represented by the numeral 1 in the tens column. Repeat for other numbers.

Using finger math (Chisenbop, see Appendix), multiplication by ten can be replicated by switching from the ones hand (the right hand) to the tens hand. In  $3 \times 10$ , for example, we start with three ones on the right hand. Each one multiplied by ten becomes a 10, so each ones finger is replaced by a tens finger on the left hand:



Older children can also use the place value mat to work on multiplying tens by tens. Place a bundle of tens in the tens column. What happens when we multiply by 10? We get ten bundles of ten, or a bundle of 100 in the hundreds column. Repeat with two or three bundles of ten.

Now that students understand that multiplying 10 by 10 equals 100, you can show that multiplying one by one hundred involves multiplying by ten twice;  $1 \times 10 = 10$  (a bundle of tens);  $10 \times 10 = 100$  (ten bundles of tens, or one bundle of 100). So multiplying by 100 involves moving from the ones to the hundreds column. Again, show how the written algorithm represents what happens on the place value chart: Multiplying one by 100 results in no ones, no tens, and one hundred. This is represented by a zero in the ones column, a zero in the tens column, and a one in the hundreds column. (Note: Avoid the language of “adding a zero”, as this is confusing. Adding a zero does not change a number:  $1 + 0 = 1$ ). You are placing a zero in the ones column, and shifting the ones to the tens, the tens to the hundreds column etc.

Once you have moved to proportional manipulatives, where a chip in the tens column represents a ten etc., you can go to multiplying 2-digit numbers by 10 or by 100. Multiplying 23 by 10, for instance, turns the two tens into two hundreds, the three ones into three tens, and leaves nothing in the ones column on the place value chart. On paper, this is reflected by the zero in the ones column.

<b>Drill 34</b>			
<b>Multiplication by 10</b>			
2x10	3x10	4x10	5x10
6x10	7x10	8x10	9x10
10x10			
28x10 325x10 etc.			

It is only after students clearly understand the underlying concept of multiplying by ten, one hundred, etc. that you should include mental arithmetic problems involving multiplication by ten and multiples of ten in your daily mental arithmetic exercises. Choose from the skills in the drill boxes below according to the level of your students.

### Multiplication by 11

There are several ways to learn multiplication by 11 without memorizing each individual result. The first one is the simplest, and is an extension of multiplying by 10. To multiply

a number by 11, multiply that number by 10, then add the number. For example:  $11 \times 7 = 10 \times 7 + 7 = 70 + 7 = 77$ . This pattern also holds for larger numbers, of course:  $11 \times 25 = 10 \times 25 + 25 = 250 + 25 = 275$ .

But there is another “trick” to the multiplication of 11, which can be illustrated by doing a long multiplication problem with 11:

$\begin{array}{r} 35 \\ \times 11 \\ \hline 35 \\ 35 \\ \hline 385 \end{array}$	<p>Note that the ones digit of the result is the ones digit of the number multiplied by 11. The tens digit is the sum of the ones and tens digits of the number multiplied by 11. Finally, the hundreds digit of the result is the tens digit of the original number. Here is another 2-digit example:</p>												
	<table style="border-collapse: collapse;"> <tr> <td style="padding-right: 20px;"><math>11 \times 43</math>.</td> <td>Ones digit:</td> <td style="text-align: right;">3</td> </tr> <tr> <td></td> <td>tens digit: <math>3+4</math></td> <td style="text-align: right;">7</td> </tr> <tr> <td></td> <td>hundreds digit</td> <td style="text-align: right;">4</td> </tr> <tr> <td></td> <td>Answer</td> <td style="text-align: right;">473</td> </tr> </table>	$11 \times 43$ .	Ones digit:	3		tens digit: $3+4$	7		hundreds digit	4		Answer	473
$11 \times 43$ .	Ones digit:	3											
	tens digit: $3+4$	7											
	hundreds digit	4											
	Answer	473											

**Drill 35**  
**Multiplication by 11**

$2 \times 11$	$3 \times 11$	$4 \times 11$	$5 \times 11$
$6 \times 11$	$7 \times 11$	$8 \times 11$	$9 \times 11$
$10 \times 10$	$11 \times 11$	$12 \times 11$	
$28 \times 11$	$75 \times 11$	$237 \times 11$	etc.

One small modification is necessary when the two digits of the number add up to more than ten. This will simply modify the hundreds digit of the result by adding an extra hundred. Example:  $11 \times 68$ . Result: Ones digit: 8; tens digit:  $6+8=14$ , keep the 4, carry the 1; hundreds digit  $6+1$ . Answer: 748

$\begin{array}{r} 435 \\ \times 11 \\ \hline 435 \\ 435 \\ \hline 4785 \end{array}$	<p>This pattern can be extended to 3-digit or larger numbers, as shown to the left. The ones digit of the result is the ones digit of the original. The tens digit is the sum of the ones and tens digits. The hundreds digit is the sum of the tens and hundreds digits. And the thousands digit is the original hundreds digit. Again, if the digits add up to more than ten, it is necessary to carry. To carry out this algorithm, you may want to</p>
---	--

encourage your students to write down the number. Here is one more example:

$843 \times 11$ :	Ones digit	3	
	Tens digit:	$4 + 3$	7
	Hundreds digit	$8 + 4$	2
	Thousands digit	$8 + 1$	9
	Result	<u>9272</u>	(carry the 1)

Multiplication by 12

**Drill 36**  
**Multiplication by 12**

$2 \times 12$	$3 \times 12$	$4 \times 12$	$5 \times 12$
$6 \times 12$	$7 \times 12$	$8 \times 12$	$9 \times 12$
$10 \times 12$	$11 \times 12$	$12 \times 12$	

Rather than using rote memory for multiplying by twelve, students should be encouraged to first multiply by ten, then add two times the number. This is the result of the distributive property of multiplication and addition:  $12 \times 8 = (10+2) \times 8 = 10 \times 8 + 2 \times 8 = 80 + 16 = 96$ . Again, this serves to reduce the facts that need to be memorized, and it creates connections that help students better understand

mathematics. In the long run, the number will be committed to memory without the intermediate steps. But even if that does not happen, the additional steps do not take very long to execute.

### Multiplying larger numbers by 2, 4, and 8

To multiply by two and by multiples of two, we use the fact that doubling a number is

#### **Drill 37** **Multiplication by 2**

$2 \times 34$	$2 \times 324$	$2 \times 4312$
$2 \times 63$	$2 \times 483$	$2 \times 5743$
$2 \times 78$	$2 \times 678$	$2 \times 3892$

easier than multiplying by larger factors. Multiplying by two (doubling) simply involves doubling each digit, while carrying when appropriate. For instance,  $2 \times 124 = 248$ : each digit is simply doubled. For larger digits, it is necessary to carry: For instance,  $2 \times 438$ : Starting with the ones digit,  $2 \times 8 = 16$  (keep the 6, carry the 1);  $2 \times 3 = 6 \rightarrow 7$  (adding the carried 1);  $2 \times 4 = 8$ . Answer: 876. Drill 37 involves doubling numbers.

Since  $4 = 2 \times 2$ , multiplying by 4 involves doubling a number twice, or multiplying by 2 twice. Students simply use the skill used in drill 37. Similarly, since  $8 = 2 \times 2 \times 2$ , multiplying a number by 8 involves doubling it three times. For example,  $4 \times 23 = 2 \times 23 \times 2 = 46 \times 2 = 92$ . And  $8 \times 23 = 2 \times 23 \times 2 \times 2 = 46 \times 2 \times 2 = 92 \times 2 = 184$ .

#### **Drill 38** **Multiplication by 4 and 8**

$4 \times 34$	$4 \times 62$	$4 \times 87$
$8 \times 34$	$8 \times 62$	$8 \times 87$ etc.

### Multiplication of Larger Numbers by 5

To multiply larger numbers by 5, we make use of the fact that  $5 = 10/2$ , and that it is easy to multiply by 10 or to divide by 2. To multiply by 5, therefore, we first multiply by 10, then divide by 2. For example,  $26 \times 5 = 26 \times 10 / 2 = 260 / 2 = 130$ . We are assuming here that students have learned to divide by 2, although we have not talked about division here. Another example:  $75 \times 5 = 75 \times 10 / 2 = 750 / 2 = 375$ . Note that if the last digit of the number multiplied by 5 is odd, the answer always has a 5 in the ones digit; if the number is even, the answer has a 0 in the ones digit. Students have learned this when learning their fives facts.

### Multiplication of Larger Numbers by 9

#### **Drill 40** **Multiplication by 9**

$9 \times 34$	$9 \times 62$	$9 \times 57$
$9 \times 136$	$9 \times 245$	$9 \times 687$ etc.

Note: Multiply by 10 and subtract number

To multiply by 9, we use the fact that  $9 = 10 - 1$ , and multiply the number by ten first, then subtract the number from the result. Again, this is using the distributive property: For example,  $9 \times 35 = (10 - 1) \times 35 = 10 \times 35 - 1 \times 35 = 350 - 35 = 315$ . We can check the result quickly: the sum of the digits in the result will add to 9 or a multiple of 9. This is true of all multiples of 9. In this case  $3 + 1 + 5 = 9$ , so we are probably correct. Another example:  $9 \times 78 = 780 - 78$

= 702. Check:  $7+2 = 9$ . Finally,  $83 \times 9 = 830 - 83 = 747$ . Check:  $7+4+7 = 18$ . This is a multiple of 9; note that if we repeat the process and add  $1+8$  we will get 9.

### Multiplication by 25

Since  $25 = 100/4$ , we can multiply by 25 by first multiplying by 100, then dividing by 4. For example,  $16 \times 25 = 16 \times 100/4 = 1600/4 = 400$ . If we have a number that is more difficult to divide by 4, we can divide by 2 twice instead:  $15 \times 25 = 15 \times 100/4 = 1500/4 = 750/2 = 375$ . Again, this assumes students have learned to divide.

### Multiplication of 2-digit Numbers

The following exercise provides a shortcut for students who have learned 2-digit multiplication. However, it also provides an introduction to this process for students who are just learning it. We are explaining the process by using the multiplication grid as a visual aid (see appendix). The activity described below illustrates the results of 2-digit multiplication, accounting for the digit in each place, ones, tens and one hundreds.

Use of the multiplication grid provides an example of the array model of multiplication, i.e. a situation where multiplication is used to compute area or volume. In the example in this text, we will be multiplying  $12 \times 13$ . The chart on the next page provides an illustration of the results of the activity.

To carry out this activity in the classroom, provide each student with the multiplication graph paper included with this appendix. Each student also needs a ruler and a pencil, as well as four coloring pens (Crayolas will be fine). For the teacher, it is useful to have a transparency of the multiplication grid to show each of the steps. Coloring pens for transparencies help, but Crayola pens will do the trick as well.

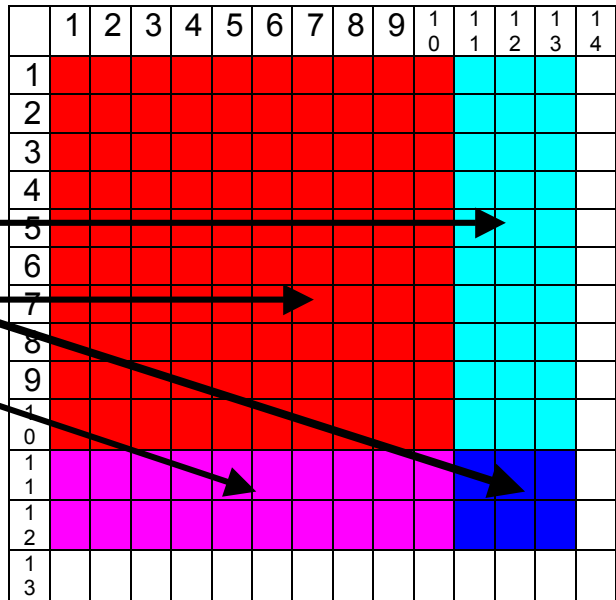
The first step is to have students mark the numbers to be multiplied in the rows and columns of the graph paper – have them place the mark on the outside of the number, i.e. to the right and bottom of the numbers to be multiplied. In our example, the students mark off 13 on the top and 12 along the vertical line. Then tell students to use a ruler to draw lines around the area that shows  $13 \times 12$ .

Next, have students identify the area representing 100, and draw a line around it. Ask them to color that area in one of the colors. Then have students identify the two areas that represent multiples of tens (two lightly shaded rectangular areas above); each area should be filled in a different color. Finally, students identify the area representing the six ones (dark corner above), and color it using a new color.

Graph representing 12 x 13

$$\begin{array}{r}
 12 \\
 \times 13 \\
 \hline
 2 \times 3 \quad 6 \\
 3 \times 10 \quad 30 \\
 2 \times 10 \quad 20 \\
 10 \times 10 \quad 100 \\
 \hline
 156
 \end{array}$$

Note the similarity to the regular multiplication algorithm. The 6 and 30 are usually combined in one line, as are the 20 and 100 (the last 0 may be represented by a placeholder). The above algorithm may be easier for some students.

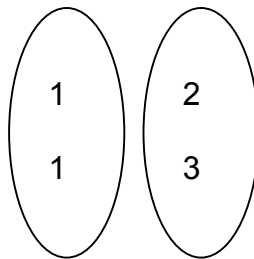


Ask students to count the squares in each of the colored areas: the ones, the two sets of tens, and the hundreds. Now show them the relation between the graph and the multiplication algorithm, as above. This activity is very useful when first introducing multi-digit multiplication.

Students who have learned multi-digit multiplication can also use this principle to do two-digit multiplication mentally.

**Drill 41**  
**2-Digit Multiplication**

45 x 23	76 x 15
37 x 21	36 x 27
etc.	



Ones:	2 x 3 = 6
Tens:	2 x 10 = 20
	3 x 10 = 30
Hundreds:	10 x 10 = <u>100</u>
	156

### Using Napier Rods in Long Multiplication

Napier rods have been used before calculators as a way to simplify long multiplication. They are used here as an aid to help students understand the place value aspects of multiplication. Napier rods can easily be made out of tongue depressors, and children can make their own or they can be made with the assistance of an aide. It is important mainly that all of the horizontal lines on the rods line up properly. The table below can also be cut and pasted on light cardboard, and rods can be cut along the heavy lines. Note that each vertical rod contains the multiplication facts for one number, with the ones in the bottom right corner, the tens in the upper left corner of each cell.

Below is a complete set of Napier rods.

	0	1	2	3	4	5	6	7	8	9
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	1	1	1	1	1
3	0	0	0	0	1	2	3	4	5	6
4	0	0	0	1	2	4	6	8	1	2
5	0	0	1	2	4	7	1	2	3	4
6	0	0	1	2	4	7	1	2	3	4
7	0	0	1	2	4	7	1	2	3	4
8	0	0	1	2	4	7	1	2	3	4
9	0	0	1	2	4	7	1	2	3	4

To multiply a multi-digit number by a one-digit number, we proceed as in the following example:  $236 \times 5$ :

1. Select the rods corresponding to the multi-digit number, in this case 2, 3 and 6, and line them up with the first rod, as illustrated below.
2. Take a piece of paper and place it under the "5" row.
3. Add the numbers in the diagonals, starting with the ones at the right.

	2	3	6
1	0	0	0
2	0	0	1
3	0	0	1
4	0	1	2
5	1	1	3

1      1      8      0

236 x 5 using Napier Rods  
 Note the correspondence between the algorithmic solution and the napier rods:  
 $5 \times 6 = 30$  (right bottom box). Write the 0, carry the 3.  
 $5 \times 3 = 15$ , add the carried 3 to the 5, write 8, carry the 1  
 $2 \times 5 = 10$ , add the carried 1 to the 0, write 1, carry the 1  
 Answer: 1,180

As can be seen from the example above, Napier rods provide a nice way to reinforce the multiplication algorithm. Note that numbers in each diagonal represent a different digit in the place value system: The diagonal to the right containing the “0” represents the units digit. The one next to it containing the “3” and the “5” represents the tens digit. The diagonal to the left of that with the “1” and “0” represents hundreds, and finally the last column containing the “1” represents the thousands column.

### Lattice Multiplication

Lattice multiplication is an extension of the logic of Napier rods, but allows the multiplication of multi-digit numbers by other multi-digit numbers. What is required is grid paper with large boxes, which is provided in the appendix. Numbers to be multiplied are written in the top row and right column of the grid, with the ones digit as the right-most digit for the number written in the top row, and the bottom digit in the right column. Lattice multiplication requires knowledge of the simple multiplication facts. Addition, as with Napier rods, is along the diagonal. As with Napier rods, each diagonal contains numbers that represent the same decimal place value.

Example:  $536 \times 729 = 390,744$

	5	3	6	
3	3 5	2 1	4 2	7
9	1 0	0 6	1 2	2
1 0	4 5	2 7	5 4	9
	17	14	4	

In the days of calculator use, lattice multiplication may not be the way students should be taught to multi-digit multiplication. However, it is an exercise that is worth doing, as it reinforces the notion of place value. The numbers in each diagonal represent the same place value, and students can verify the results within each cell. In the cell in the bottom right corner, we have the result of the multiplication of the two ones digits,  $6 \times 9$ , or 54. The “4” represents ones, the “5” represents tens. Above that cell is the result of multiplying the tens digit (2) of the number 729 by the ones digit (6) of the number 536. The cell “12” in that cell represents 120, the result of multiplying  $6 \times 20$ .

For those interested in pursuing lattice multiplication with their students, the appendix contains a sheet that can be used for this purpose.

## Division

The table below is, of course, the multiplication table. Since division is the reverse of multiplication, the multiplication table can be used to practice division as well. This table is included in the appendix, and can be used by students to practice division facts. Ask students to use a piece of paper or a 5x7 note card with an arrow in one corner. Use the table as illustrated below: The paper is placed so the arrow points at the dividend (72), and the paper is aligned with the divisor (the lower end, between 8 and 9 in this case). The other edge of the paper will now point to the answer. The lightly shaded area in the table below represents the paper that is used as the pointer.

**Division Table**

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60

$72 : 9 = 8$

$$72 : 8 = 9$$

<b>7</b>	<b>7</b>	<b>14</b>	<b>21</b>	<b>28</b>	<b>35</b>	<b>42</b>	<b>49</b>	<b>56</b>	<b>63</b>	<b>70</b>
←—————→										
<b>8</b>	<b>8</b>	<b>16</b>	<b>24</b>	<b>32</b>	<b>40</b>	<b>48</b>	<b>56</b>	<b>64</b>	<b>72</b>	<b>80</b>
<b>9</b>	<b>9</b>	<b>18</b>	<b>27</b>	<b>36</b>	<b>45</b>	<b>54</b>	<b>63</b>	<b>72</b>	<b>81</b>	<b>90</b>

1	10	20	30	40	50	60	70	80	90	100
0										

Unfortunately, the commutative property that applies to multiplication does not apply to division, so we cannot simply erase half of the table as we did in multiplication (because, for instance,  $6 \times 8 = 8 \times 6$ ). However, as the example above shows, there are number pairs in multiplication that students should be made aware of. There are three numbers involved in division: The dividend (the number to be divided), the divisor (the number to be divided by), and the quotient (the result). For any given dividend, the divisor and the quotient are reversible. For instance,  $35 : 7 = 5$ ; or  $35 : 5 = 7$ . This may help students reduce the number of facts they have to memorize.

Combining the teaching of multiplication and division will also help students, as it will make it easier for them to link division facts with the multiplication facts they have

<b>Drill 42</b>				
<b>Find Multipliers for Product</b>				
What 2 numbers are multiplied to get this product:				
8	12	27	35	42
56	64	72	81	90
etc.				

already learned. When students connect the product 56 with  $7 \times 8$ , for instance, it will be easier for them to remember that  $56 : 7 = 8$ , and that  $56 : 8 = 7$ .

To teach division, therefore, it is easiest to start by giving students a product, and to recall with numbers multiplied by each other result in that product. Note that there is sometimes more than one answer:  $16 = 2 \times 8$  or  $4 \times 4$ . You may ask

students to find other products that result from the multiplication of different pairs of numbers as an exercise (other examples: 12, 18, 24 etc.).

There is another way to link multiplication to division, and that is as a way of checking the results of a division. If  $64 : 8 = 6$ , then  $6 \times 8 = 64$ . This is useful in itself, as a way for students to check their own answers. But it is also another way to tie something that is new to something that is already known.

We will now proceed to explore tricks to assist students with dividing by each of the numbers up to 10.

Division by 2

For numbers up to 20, students must memorize the division facts. This should be easy,

<b>Drill 43</b>				
<b>Division by 2</b>				
2:2	4:2	6:2	8:2	10:2
12:2	14:2	16:2	18:2	20:2

as they have worked on doubles in multiplication. There are 10 numbers to remember, from  $2 : 2$  to  $20 : 2$ . Both of those numbers are easy, though they must be taught. This leaves 8 additional numbers:  $4 : 2$ ,  $6 : 2$  etc. These numbers should not be hard to memorize.

For larger 2-digit dividends, division by 2 is also simple. Let us first divide numbers in which the tens digit is an even number. We divide the number in to two parts, the tens and the ones, and divide each part separately, then add the results together. For example,  $46 : 2 = 40 : 2 + 6 : 2 = 20 + 3 = 23$ . The same procedure works for larger numbers when all digits are even:  $842 : 10 = 800 : 2 + 40 : 2 + 2 : 2 = 400 + 20 + 1 = 421$ . Note that we are using the distributive property of division here:  $(40 + 6) : 2 = 40 : 2 + 6 : 2$ .

**Drill 43**  
**Division by 2**

24:2   36:2   68:2   84:2

32:2   54:2   78:2   96:2

Next, we deal with 2-digit numbers in which the tens digit is an odd number. Again, we divide the number into two components, but we take one of the tens and join it with the ones to obtain an even tens digit. For example,  $76 = 60 + 16$ ;  $76 : 2 = 60 : 2 + 16 : 2 = 30 + 8 = 38$ .

Again, this principle can be extended to larger numbers. To divide 352 by 2 we divide 352 as follows:  $352 = 200 + 152 = 200 + 140 + 12$ . Now we divide each component by 2, and add the results:  $100 + 70 + 6 = 176$ . Note that  $140 : 2 = 14 : 2 \times 10$ .

Divisibility by 2

A number is divisible by two if it is an even number, that is if its units digit is a 0, 2, 4, 6, or 8.

Division by 3

Again, there is no real shortcut to dividing numbers by three than to memorize the series, from  $3 : 3$  to  $30 : 3$ . Again, these two are easy, and there are 8 other numbers to memorize. The best approach is to first review products, to have students link multiplication and division. For instance, 21 is the product of which two numbers?  $21 = 7 \times 3$ . So  $21 : 3 = 7$ , etc.

**Drill 44**  
**Division by 3**

3:3   6:3   9:3   12:3   15:3

18:3   21:3   24:3   27:3   30:3

42:3   54:3   72:3   81:3   96:3

126:3   231:3   411:2   621:3

For larger numbers, divide the number into components that are divisible by 3, divide each component by 3, and add the results, as we did in division by 2. Example:  $51 : 3 = 51 = (30 + 21) / 3 = 10 + 7 = 17$ . Example:  $135 : 3 = 120 + 15 = 40 + 5 = 45$ . Example:  $522 : 3$ ;  $522 = 300 + 222 = (300 + 210 + 12) / 3 = 100 + 70 + 4 = 174$ .

Divisibility by 3

A number is divisible by 3 if the sum of its digits adds to a multiple of 3. Check the numbers in drill box 44 in the last row:  $126 \rightarrow 1 + 2 + 6 = 9$ . Since 9 is a multiple of 3, 126 is divisible by 3;  $621 \rightarrow 6 + 2 + 1 = 9$ ; 621 is divisible by 3 since 9 is a multiple of 3.

## Division by 4

### **Drill 45** **Division by 4**

4:4    8:4    12:4    16:4    20:4  
24:4    28:4    32:4    36:4    40:4

Note: To divide by 4, divide by 2 twice.

76:4    52:4    88:4    96:4    104:4

Proceed as with division by 2 and by 3, and start with products. Ask students to recall the two factors associated with each of the products of 4 times another number up to 10. Besides 4 and 40, students will have to recall 8 products, and the 8 associated division facts.

There is another way to learn division by 4, especially if students have mastered division by 2. Since  $4 = 2 \times 2$ , instead of dividing by 4 we

can divide by 2 twice. For instance,  $24:2 = (24:2):2 = 12:2 = 6$ . Similarly for larger numbers,  $72:4 = (72:2):2 = 36:2 = 18$ .

## Divisibility by 4

A number is divisible by 4 if it is divisible by 2 twice. For numbers greater than 100, it is only necessary to check the tens and ones digits, as 100 is divisible by 4, and so are all its multiples. In practical terms, it is only necessary to divide by 2 once, and then check to see if the result is even. If it is, the number is divisible by two twice, and thus divisible by 4. If the result of the first division by 2 is an odd number, the number is not divisible by 4.

Examples:       $82 : 2 = 41$ . This is an odd number, 82 is not divisible by 4.  
                     $96 : 2 = 48$ . This is divisible by 2, so 96 is divisible by 4.  
                     $2572 \rightarrow 72 : 2 = 36$ . This is divisible by 2, so 2572 is divisible by 4.

## Division by 5

To prepare for division by 5, again have students recall the factors needed to get a given product of 5. For instance,  $35 = 5 \times 7$ . So  $35 : 5 = 7$ . There is also an easy trick to remembering division by 5, which comes from the fact that  $5 = 10 : 2$ . This means that, to divide a number by 5, we can multiply it by 2, then divide by 10.

Examples:       $25 : 5 = (25 \times 2) : 10 = 50 : 10 = 5$   
                     $45 : 5 = (45 \times 2) : 10 = 90 : 10 = 9$   
                     $40 : 5 = (40 \times 2) : 10 = 80 : 10 = 8$   
                     $245 : 5 = (245 \times 2) : 10 = 490 : 10 = 49$

In fact, we are dividing by 10, which is twice as much as what we want to divide by (5), so we have to make up for it by multiplying by 2. It is easier to multiply by 2 first, because it avoids getting decimals. Otherwise, the order is not important, we could divide by 10 first, then multiply by 2. The

### **Drill 46** **Division by 5**

5:5    10:5    15:5    20:5    25:5  
30:5    35:5    40:5    45:5    50:5

Note: To divide by 5, multiply by 2 and divide by 10.

125:5    315:5    420:5    624:5

result is the same. This trick can also easily be applied to larger numbers.

### Divisibility by 5

A number is divisible by 5 if its ones digit is either 0 or 5.

### Division by 6

Again, start with products of 6 and have students recall the factors which result in each product. For 42, the factors are 6 and 7,  $6 \times 7 = 42$ . So  $42 : 6 = 7$ . There is another way to divide by 6, since  $6 = 2 \times 3$ . To divide by 6, first divide by 2, then by 3.

Example:  $24 : 6 = (24 : 2) : 3 = 12 : 3 = 4$

$42 : 6 =$

**Drill 47**  
**Division by 6**

6:6   12:6   18:6   24:6   30:6  
36:6   42:6   48:6   54:6   60:6

Note: To divide by 6, divide by 2, then divide by 3.

66:6   78:6   84:6   90:6   108:6

$(42 : 2) : 3 = 21 : 3 = 7$

### Divisibility by 6

A number is divisible by 6 if it is divisible by 2 and by 3. In practical terms, a number is divisible by 6 if it is even (i.e. divisible by 2), and if the sum of its digits are divisible by 3.

Examples: 243 → odd number, not divisible by 2, therefore not divisible by 6

246 → even number, digits add up to 12; divisible by 2 and 3, and by 6

248 → even number, digits add up to 14, not divisible by 3 and not by 6

### Division by 7

Nothing will do here but rote memorization. There are no tricks, except tying division by 7 to multiplication, as we showed with the other numbers. Provide students with products where one of the factors is 7, and have them recall the corresponding multiplication facts. Then drill the division facts related to 7. Example  $56 = 7 \times [?]$ . Since  $56 = 7 \times 8$ ,  $56 : 7 = 8$ .

**Drill 47**  
**Division by 7**

7:7   14:7   21:7   28:7   35:7  
42:7   49:7   56:7   63:7   70:7

Use the multiplication/division chart in the appendix so students can practice on their own, at school as well as at home.

### Divisibility by 7

There is a divisibility check for 7, but it is more complicated than simply dividing a number by 7. So we won't reproduce it here, it simply is not worth it.

### Division by 8

Besides connecting division by 8 with the known multiplication facts as shown for the other numbers, there is an alternative way to divide by 8. Since  $8 = 2 \times 2 \times 2$ , to divide by 8 we must divide a number by 2 three times.

#### Examples

### Divisibility by 8

A number is divisible by 8 if it is divisible by 2 three times. Since 1000 and its multiples are divisible by 8, we only need to check the last three digits of a whole number, the hundreds, tens and ones. In practical terms, a number is divisible by 8 if we get an even number after dividing by 2 twice.

Examples: 481  $\rightarrow$  Odd number, not divisible by 8  
482  $\rightarrow$   $482:2 = 241$ ; odd number, not divisible by 8  
484  $\rightarrow$   $484:2 = 242$ ;  $242:2 = 121$ ; odd number, not divisible by 8  
488  $\rightarrow$   $488:2 = 244$ ;  $244:2 = 122$ ; even number, divisible by 8  
1,488  $\rightarrow$  check 488, as proceed as above

#### **Drill 48** **Division by 8**

8:8    16:8    24:8    32:8    40:8  
48:8    56:8    64:8    72:8    80:8

Note: To divide by 8, divide by 2 three times.

168:8    256:8    448:8    648:8

### Division by 9

Besides relating division by 9 to the multiplication facts that are already known, there is another way to divide by 9. Since  $9 = 3 \times 3$ , to divide by 9 we can divide by 3 twice. Since students already have learned to divide by 3, this can be used when students have not achieved instant recall of the division facts relating to the number 9. This will require reviewing the multiplication of 3 beyond 10, but this is not hard. It is simply necessary to divide a number into multiples of 30 and remainders. For instance,  $36 : 3 = 30:3 + 6:3 = 12$ .  $72 : 3 = 60:3 + 12:3 = 30:3 + 30:3 + 12:3 = 20 + 4 = 24$ .

#### **Drill 49** **Division by 9**

9:9    18:9    27:9    36:9    45:9  
54:9    63:9    72:9    81:9    90:9

Note: To divide by 9, divide by 3 twice.

126:9    153:9    261:9    342:9

Example:  $36 : 9 = (36 : 3) : 3 = 12 : 3 = 4$ .  
 $117 : 9 = (117 : 3) : 3 = 39 : 3 = 13$  [ $117 : 3 = 90 : 3 + 27 : 3 = 30 + 9 = 39$ ]

### Divisibility by 9

A number is divisible by 9 if the sum of its digits is divisible by 9. Examples:  $27 \rightarrow 2+7 = 9$ , 27 is divisible by 9;  $342 \rightarrow 3+4+2 = 9$ , 342 is divisible by 9;  $981 \rightarrow 9+8+1 = 18$ , 18 is divisible by 9, 981 is divisible by 9. Another way to do this, especially when the sum is greater than 9, is to cancel nines from the digits; the result should then be 0. Example:  $72,981 \rightarrow$  cancel  $7+2=9$ ; cancel 9; cancel  $8+1$ . Remainder is 0, 72,981 is divisible by 9.

**Drill 50**  
**Divisibility**

Check the following numbers to see if they are divisible by 2, 3, 4, 5, 6, 8, or 9:

Division by 10 and Multiples of 10

The appendix contains a place value chart that can be used to illustrate division by 10 and multiples of 10.

Other shortcuts in Division

Division by 50

To divide by 50, multiply by 2 and divide by 100.

Division by 25

To divide by 25, multiply by 4 and divide by 100.